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A THEORY OF INDIVIDUAL

CHOICE BEHAVIOR

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1. Introduction

One large portion of psychology is segmented — even as chapter headings in many texts — into such topics as sensation, motivation, simple selective learning, reaction time, etc. Unlike another broad class of psychological problems (e.g., memory, thinking and perception), these areas have a common theme: choice. To be sure, in the study of sensation the choice is between stimuli, in learning it is between responses, and in motivation between alternatives having different preference evaluations; and some psychologists believe that these distinctions, at least the one between stimulus and response, are basic to an understanding of behavior. This paper will attempt a mathematical description of individual choice behavior where the distinction is not made except in the language used in different interpretations of the theory. Thus, I shall use the more neutral word "alternative" to include the several cases. There seems

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little point in trying to discuss the merits and demerits of this decision now, except to mention it in order to avoid confusion later. The results that follow — which I believe afford some insight into, and some integration of, utility theory, psychological and psychophysical scaling, and learning theory — will implicitly serve as the argument for the course taken.

A basic presupposition of the paper is that choice behavior is probabilistic, not algebraic. This is a commonplace in psychology, but it is a comparatively new and unproven point of view in economics (utility theory). To be sure, economists when pressed will admit that the psychologist's assumption is probably the more accurate, but they have argued that the resulting simplicity warrants an algebraic idealization. Ironically, some of the following results suggest that, on the contrary, the idealization may actually have made their problem artificially difficult.

Once the probabilistic nature of choice behavior is admitted, a problem arises which does not exist in the algebraic models. Complete data as to what choices a person will make from each possible pair of alternatives do not appear to determine what choice he will make when there are three or more alternatives from which to choose. Because they cannot escape multiple alternative choice problems, economists have been particularly sensitive to this feature of probabilistic models and it has undoubtedly been one of the sources of resistance in their admitting imperfect discrimination. Early psychologists, particularly learning theorists experimentally studied multiple alternatives, but as the data seemed dreadfully complicated a trend set in toward fewer and fewer alternatives until now most

studies are conducted on simple T mazes. One can say that, for the most part, present day psychologists have been willing to ignore — or, to be more accurate, to bypass and postpone — the connections between pairwise choices and more general ones. And so the relationships have remained obscure.

I shall center my attention upon this problem. The method of attack is to introduce a single axiom relating the various probabilities of choices from different finite sets of alternatives. It is a simple and, I feel, intuitively compelling axiom that appears to illuminate many of the more traditional problems, in particular the question as to whether or not a comparatively unique numerical scale exists and reflects choice behavior. Such a scale, unique except for its unit, will be shown to exist very generally. It appears to be the formal counterpart of the intuitive idea of utility (or value) in economics, of incentive value in motivation, of subjective sensation in psychophysics, and of response strength in learning theory.

2. The Basic Axiom

Throughout the paper I shall suppose that a universal set U is given which is to be interpreted as the universe of possible alternatives (stimuli or responses). In practice, U will have to possess a certain homogeneity: the decision maker will have to be able to evaluate the elements of U according to some comparative dimension and to be able to select from certain finite subsets the elements that he thinks are superior (or inferior, or distinguished in some way) along that dimension. For example, in economics U may be taken to be a finite set of goods among which a person can express preferences;

in psychophysics, it may be the infinite set of possible sound energies (at a fixed frequency) which a subject can be asked to evaluate as to loudness; or, in learning theory, U may be the set of alternative responses available to the organism. Note that U may be finite or infinite.

In general, a subject is not asked to make a choice from the whole of U, but rather from some finite subset. In a great many experiments only two stimuli are presented to the subject at a time and he is required to choose the one he prefers, or the one he deems louder, etc. Of course, larger subsets could be used, although for the most part they have not been, and certainly most daily decisions are from larger subsets (e.g., the choice of a meal from a menu or the choice between several jobs, etc.).

Let T be a finite subset of U and suppose that the choice of an element must be confined to T. If S is a subset of T, let P(S;T) denote the probability that the selected element is a member of the subset S when the choice is restricted to T. These probabilities will be the basic ingredients of the following theory. It should be emphasized that P(S;T) denotes the probability that the single chosen element lies in S, not the probability that S is chosen. If, however, $S = \{x\}$, i.e., S is the single element set having x as its only member, then $P(\{x\};T)$ denotes the probability that x is selected when T is offered. I shall abbreviate this to P(x;T). If $T = \{x,y\}$ then $P(x;\{x,y\})$ is written simply as P(x,y) and it denotes the probability that x is chosen over y when only x and y are offered. The symbol P(x,x) will be admitted as meaning $P(x;\{x,y\})$ where y = x, not $P(x;\{x\})$, and by convention it will always have the value 1/2.

A bit more formally, we shall suppose that for certain finite[2] subsets T of U a probability measure is given on the subsets of T which, by the axioms of probability theory, satisfies:

i. for $S \subset T$, $0 \leq P(S;T) \leq 1$,

ii. $P(T;T) = 1$,

iii. if $R, S \subset T$ and $R \cap S = \phi$, then

$$P(R \cup S;T) = P(R;T) + P(S;T).$$

Repeated application of iii implies that

$$P(S;T) = \sum_{x \varepsilon S} P(x;T).$$

Note, given our interpretation of these probabilities, part ii means that the subject is forced to make a choice: the probability is 1 that his choice is in T when only T is available.

The axiom that we shall explore is this:

<u>Axiom 1</u>. <u>If</u> T <u>is finite and</u> $R \subset S \subset T$, <u>then</u> $P(R;T) = P(R;S)P(S;T)$.

At first glance, the axiom may seem to be a tautology, but it is not; and, at a second glance, it may seem unlikely. So some discussion is needed. Of course, it would be a tautology if the quantity P(R;S) were replaced by the conditional probability that the choice is in R when it was made from T and is known to lie in S. But P(R;S) is only the probability that the choice is in R when it must be made from S, not from T. Thus, the axiom has content. As an example, suppose that T is the set of entrees on a certain menu, S the set of meat dishes, and R the set of beef dishes. Then P(R;T) denotes the probability that the chosen entree is a beef dish when the whole menu

2. The restriction to finite sets is not basic, but as that is all that is needed for most of the following results, and as the theory is easier to state for that case, I shall assume them to be finite except in section 8.

is presented; $P(S;T)$ the probability that it is a meat dish when the whole menu is available; and $P(R;S)$ the probability that it is a beef dish when only the meat dishes are presented (as sometimes happens late in the evening when various items have run out).

In decision theory (see, for example, [8]) one axiomatic idea is recurrent: the independence of irrelevant alternatives. It is captured differently depending upon the context, but the common nature of these axioms is clear. Intuitively, the idea is that if you are to make a choice from a set of alternatives, then the addition of new alternatives which are clearly inferior to your original choice should not cause you to alter that choice. In a slightly more subtle form, this is the content of axiom 1. It says that if, for whatever reason, attention has been confined to S, then the existence of the alternatives in T-S cannot influence the probability that the choice is in R.

I shall postpone any discussion of the empirical plausibility of this axiom until we know some of its consequences (see section 11).

Before presenting any results, let me recast the axiom in several alternate forms. Define

$$Q(S;T) = 1-P(S;T).$$

In our interpretation, $Q(S;T)$ is the probability that the chosen element is not in S when T is the set of possible choices.

<u>Axiom 1'</u>. <u>If</u> T <u>is finite</u>, $R,S \subset T$, <u>and</u> $R \cap S = \phi$, <u>then</u> $Q(R \cup S;T) = Q(R;T-S)Q(S;T)$.

<u>Axiom 1"</u>. <u>If</u> T <u>is finite</u>, $R,S \subset T$, <u>and</u> $R \cap S = \phi$, <u>then</u> $P(R;T) = P(R;T-S)Q(S;T)$.

<u>Axiom 1"'</u>. <u>If</u> T <u>is finite</u>, $x,y \in T$, <u>and</u> $x \neq y$, <u>then</u> $P(x;T) = P(x;T-\{y\})Q(y;T)$.

The last axiom appears to be a good deal weaker than the other three, and some people find it easier to accept; however, our first result shows that all four are really the same.

Theorem 1. Axioms 1, 1', 1", and 1"' are equivalent.

Proof. 1 implies 1'. Suppose $R, S \subset T$ and $R \cap S = \phi$. Let $S' = T-S$.

Since $R \subset S'$,

$$\begin{aligned}
P(R;T) &= P(R;S')P(S';T) && \text{(axiom 1)} \\
&= P(R;S')[1-P(T-S';T)] && \text{(ii and iii)} \\
&= P(R;S')[1-P(S;T)] && \text{(def. of } S'\text{)} \\
&= P(R;S')Q(S;T) && \text{(def. of } Q\text{).} \qquad (1)
\end{aligned}$$

Consider,

$$\begin{aligned}
Q(R \cup S;T) &= 1-P(R \cup S;T) && \text{(def. of } Q\text{)} \\
&= 1-P(R;T)-P(S;T) && \text{(iii)} \\
&= Q(S;T)-P(R;S')Q(S;T) && \text{(def. of } Q \text{ and eq. 1)} \\
&= Q(S;T)[1-P(R;T-S)] && \text{(def. of } S'\text{)} \\
&= Q(S;T)Q(R;T-S) && \text{(def. of } Q\text{).}
\end{aligned}$$

1' implies 1". Consider,

$$\begin{aligned}
P(R;T-S)Q(S;T) &= Q(S;T)-Q(R;T-S)Q(S;T) && \text{(def. of } Q\text{)} \\
&= Q(S;T)-Q(R \cup S;T) && \text{(axiom 1')} \\
&= P(R \cup S;T)-P(S;T) && \text{(def. of } Q\text{)} \\
&= P(R;T) && \text{(iii).}
\end{aligned}$$

1" implies 1. Let $R \subset S \subset T$ and call $S' = T-S$, then

$$\begin{aligned}
P(R;T) &= P(R;T-S')Q(S';T) && \text{(axiom 1")} \\
&= P(R;S)[1-P(T-S;T)] && \text{(def. of } S' \text{ and } Q\text{)} \\
&= P(R;S)P(S;T) && \text{(ii and iii).}
\end{aligned}$$

1" implies 1"'. Let $R = \{x\}$ and $S = \{y\}$.

1"' implies 1". We prove this by induction on $|S|$. Suppose $S = \{y\}$, then

$$\begin{aligned}
P(R;T) &= \sum_{x \varepsilon R} P(x;T) && \text{(iii)} \\
&= \sum_{x \varepsilon R} P(x;T-\{y\})Q(y;T) && \text{(axiom 1"')} \\
&= P(R;T-\{y\})Q(y;T) && \text{(iii).}
\end{aligned}$$

Let $R \cap S = \phi$ and suppose the assertion is true for all sets having $|S|-1$ or fewer elements. Let $y \in S$. Clearly, $R \cap (S-\{y\}) = \phi$, so

$$P(R;T) = P[R;(T-S)\cup\{y\}]Q(S-\{y\};T) \qquad \text{(induction hypothesis)}$$
$$= P(R;T-S)Q[y;(T-S)\cup\{y\}]Q(S-\{y\};T) \qquad \text{(axiom 1''')}.$$

However,

$$Q[y;(T-S)\cup\{y\}]Q(S-\{y\};T) = \{1-P[y;(T-S)\cup\{y\}]\}Q(S-\{y\};T) \quad \text{(def. of Q)}$$
$$= Q(S-\{y\};T)-P(y;T) \quad \text{(induction hypothesis)}$$
$$= 1-P(S-\{y\};T)-P(y;T) \quad \text{(def. of Q)}$$
$$= 1-P(S;T) \quad \text{(iii)}$$
$$= Q(S;T) \quad \text{(def. of Q)}.$$

Throughout the rest of the paper I will refer just to axiom 1, but will use whichever of the four forms is convenient.

The next theorem is basic to all later results. It states, in effect, that if axiom 1 holds, then the pairwise probabilities determine all others.

<u>Theorem 2</u>. <u>Suppose</u> $T \subset U$ <u>is finite and that</u>, for all $R \subset S \subset T$, $P(R;S)$ <u>is defined</u>. <u>If axiom 1 holds</u>, <u>then for every</u> $x \in S$,

$$P(x;S) = \frac{1}{\sum_{y \in S} \frac{P(y,x)}{P(x,y)}}$$

Proof. If S has only one element, the theorem is trivial, so we assume S has two or more elements. By axiom 1 and property iii of probability,

$$P(x;S) = P(x;\{x,y\})P(\{x,y\};S)$$
$$= P(x,y)[P(x;S) + P(y;S)],$$

for $y \in S-\{x\}$. Rewriting,

$$P(y,x)P(x;S) = [1-P(x,y)]P(x;S)$$
$$= P(x,y)P(y;S). \qquad (2)$$

There are two cases to be considered.

1. For some $y \in S-\{x\}$, $P(x,y) = 0$. Since $P(y,x) = 1-P(x,y) = 1$, this together with eq. 2 implies that $P(x;S) = 0$, and so the assertion is true.

2. For all $y \in S-\{x\}$, $P(x,y) \neq 0$. We show $P(x;S) \neq 0$. Suppose, on the contrary, $P(x;S) = 0$, then by eq. 2 $P(y;S) = 0$ for all $y \in S-\{x\}$. Thus, $P(S;S) = \sum_{y \in S} P(y;S) = 0$, which is impossible by property ii of the probability axioms, so $P(x;S) \neq 0$. This means that eq. 2 can be rewritten as

$$\frac{P(y,x)}{P(x,y)} = \frac{P(y;S)}{P(x;S)}, \tag{3}$$

for $y \in S-\{x\}$. But, by definition of $P(x,x)$, eq. 3 also holds for $y = x$, so

$$\frac{1}{\sum_{y \in S} \frac{P(y,x)}{P(x,y)}} = \frac{1}{\sum_{y \in S} \frac{P(y;S)}{P(x;S)}}$$

$$= \frac{P(x;S)}{\sum_{y \in S} P(y;S)}$$

$$= P(x;S).$$

3. Existence of a Numerical Scale

As the study of choice behavior has developed, both in psychology and in economics, one of the central problems that a formal characterization must solve is what conditions insure the existence of a relatively unique numerical scale that, in some sense, represents the choice behavior of the subjects. Mathematically, the problem is simply one of imposing sufficient axiomatic structure to prove the existence of a scale which is unique up to some group of transformations — the group of positive linear transformations (zero and unit unspecified) has usually been deemed to be just acceptable. These are what Stevens [14] calls interval

scales. But the empirical side-condition that these mathematical assumptions must form a more or less plausible description of human and animal choice behavior has rendered the problem difficult. There have been, I should say, three main approaches.

Economics. Preference among goods has been taken to be the underlying primitive, which, as an idealization, has been assumed to be an algebraic ordering of the goods. So long as a finite set of goods forms the set of alternatives, many numerical order preserving scales exist, but their uniqueness properties are completely inadequate. That being so, economists finally arrived at the position that it is safer to work only with orderings — as they say, with ordinal utilities in contrast to cardinal[3] ones — and for many of the traditional theorems of economics this is sufficient. Nonetheless, some work, particularly in modern decision theory, requires cardinal utility scales. Some extension of the traditional formulation was needed, and a decade ago it was effected by von Neumann and Morgenstern [18]. (Actually, Ramsey [11] suggested much the same idea a good deal earlier, but the importance of his work was largely missed until very recently.) Roughly, one continues to suppose that preferences are algebraic, but the domain of choice is extended from a finite set of goods to the infinite set of all possible gambles that can be generated from the goods and an infinite set of chance events. Preference over these gambles is assumed to meet certain fairly rigid axioms which, although normatively compelling, seem, at best, to lack detailed descriptive realism. Under these conditions, a scale is shown to exist which is unique up to positive linear transformations and which has the important property that the utility of a gamble is equal to the expected utility of its components.

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3. Psychologists would call this an interval scale of utility.

Psychophysics. The psychologist has been largely unwilling to make the economist's algebraic idealization, for in some measure the substance of his choice problem resides in the fact that people are unable to make consistent discriminations. The early psychophysicists proposed to use these data as a means of scaling subjective sensation. Ultimately I shall want to discuss this question more fully, mainly because recent workers have tended to reject the earlier ideas, but here I shall only point to the fact that the attempt was made and that analytical methods were presented for determining an interval scale whenever certain consistencies are exhibited by the data. Mathematically, the uniqueness of these scales results in large part from the assumption that the set being scaled is a continuum — a reasonable assumption for such dimensions as sound energy, or weight, or length, etc. For a modern discussion of this mathematics, see [7].

Psychometrics. In the remainder of psychology, a small group of workers, often referred to as psychometricians, have been concerned with scaling objects other than the traditional sensory stimuli. In particular, such concepts as attitude, preference, intelligence, and interest have concerned them. Their problem has been similar to that confronted by the economists in that their sets of alternatives are decidedly finite. Thus, the continuous approximation of the psychophysicist was out, and the gambles of the utility theorist — which, in any event, are of dubious realism in many psychological contexts — were not thought of. The resolution arrived at during the second and third decades of this century was roughly this. The, by then somewhat tarnished, psychophysical assumption was taken over that the underlying scale has the property that it makes discrimination uniform throughout the scale. Since the continuum assumption could not be transferred, this was quite insufficient to lead to a unique scale. Other assumptions

had to be added. At the time, statistics was rapidly becoming the somewhat overworked handmaiden of psychology, and normality and independence assumptions were in the wind. With little justification beyond convenience and need, these were freely introduced until finally adequate uniqueness was achieved. The result: an extensive and unsightly literature which has been largely ignored by outsiders, who have correctly condemned the _ad hoc_ nature of the assumptions.

In the other areas of choice behavior, specifically motivation and learning, it has been generally assumed that scaling or measurement is either irrelevant or can be indefinitely postponed. Among the exceptional sorties are the papers of Hull _et al_ [5] and Young [19]. However, to one familiar with measurement ideas, the notions of incentive value and response strength are suggestive of scales.

In all of the fields where scales have been important, they have been constructed under the assumption that only data for pairs of stimuli are known. In the economic models this has not been a limitation because of their algebraic nature (in particular, the assumed transitivity of preference). In the psychological models, where discrimination is admittedly not perfect, the pairwise data have not been known to determine choices from larger sets and the whole problem has remained unresolved. As we have seen (theorem 2), axiom 1, if accepted, justifies complacency on that score.

What I propose to show in this section is that if choice discrimination is admitted to be imperfect and if axiom 1 is assumed, then a scale, which is unique except for its unit, is determined for both finite and infinite sets of alternatives.[4] As we shall see, this formulation solves all of the classical problems in a very simple way.

4. In Stevens' terminology, a ratio scale will be shown to exist.

Lemma 1. *Suppose* $T = \{x,y,z\} \subset U$ *and that, for all* $R \subset S \subset T$, $P(R;S)$ *is defined, axiom* 1 *holds, and* $P(s,t) \neq 0$ *or* 1 *for* $s,t \in T$. *Then,*

$$P(x,y)P(y,z)P(z,x) = P(x,z)P(z,y)P(y,x).$$

Proof: Since $P(s,t) \neq 0$ or 1 for $s,t \in T$, eq. 2, in the proof of theorem 2, can be written

$$\frac{P(y,x)}{P(y,T)} = \frac{P(x,y)}{P(x,T)}$$

So, by theorem 2,

$$P(y,x)\left[1 + \frac{P(x,y)}{P(y,x)} + \frac{P(z,y)}{P(y,z)}\right] = P(x,y)\left[1 + \frac{P(y,x)}{P(x,y)} + \frac{P(z,x)}{P(x,z)}\right]$$

which, when simplified, yields the assertion.

Translated into words, the lemma asserts that if axiom 1 holds and if pairwise data are obtained, the probability of finding the intransitivity x,y,z,x (x "larger than" y, y "larger than" z, and z "larger than" x) is exactly the same as that of finding the reverse intransitivity x,z,y,x.

Theorem 3. *Suppose* $T \subset U$ *is finite and that, for all* $R \subset S \subset T$, $P(R;S)$ *is defined, axiom* 1 *holds, and* $P(s,t) \neq 0$ *or* 1 *for* $s,t \in T$. *There exists a non-negative real-valued function* v *on* T, *which is unique up to multiplication by a positive constant, such that*

$$P(x;S) = \frac{v(x)}{\sum_{y \in S} v(y)}$$

Proof. Let a be an arbitrary, but fixed, element of T. Define

$$v(x) = kP(x,a)/P(a,x),$$

where k is any fixed positive constant. By lemma 1,

$$\frac{P(x,y)}{P(y,x)} = \frac{P(x,a)}{P(a,x)} \Big/ \frac{P(y,a)}{P(a,y)}$$

$$= \frac{v(x)/k}{v(y)/k}$$

$$= \frac{v(x)}{v(y)} .$$

So, by theorem 2,

$$P(x;S) = \frac{1}{\sum\limits_{y\varepsilon S} \frac{P(y,x)}{P(x,y)}}$$

$$= \frac{1}{\sum\limits_{y\varepsilon S} \frac{v(y)}{v(x)}}$$

$$= \frac{v(x)}{\sum\limits_{y\varepsilon S} v(y)} .$$

If v and v' are two such functions, choose k so that $\sum\limits_{y\varepsilon S} v(y) = k \sum\limits_{y\varepsilon S} v'(y)$. Since $v \geq 0$ and $v' \geq 0$, $k > 0$. Then, for any $x\varepsilon S$,

$$P(x,S) = \frac{v'(x)}{\sum\limits_{y\varepsilon S} v'(y)}$$

$$= \frac{kv'(x)}{k \sum\limits_{y\varepsilon S} v'(y)}$$

$$= \frac{kv'(x)}{\sum\limits_{y\varepsilon S} v(y)}$$

$$= \frac{v(x)}{\sum\limits_{y\varepsilon S} v(y)}$$

So, $v(x) = kv'(x)$, for $x\varepsilon S$, which shows that v is unique up to multiplication by a positive constant.

The limitation that $P(s,t) \neq 0$ or 1 may seem severe in that it makes the scale v extremely local; however, in general this should prove to be no obstacle. One can suppose (at least sometimes, but section 10 suggests not always) that none of the probabilities are actually 0 and 1, but rather that some are very small or very large and seem to be 0 or 1 because the number of observations is finite. Can it be that these exceptional inversions are the ones we try to explain away by saying that we were not "paying attention", or we were "bored and wanted to see what would happen", etc? Nevertheless, in practice,

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we cannot estimate these limiting probabilities sufficiently accurately to use them for scaling purposes. It appears that, to construct a scale over the whole of U, one will have to find a series of overlapping T's which span U in such a way that within each T the pairwise probabilities are decidedly larger than 0 and smaller than 1. Then theorem 3 can be applied within each T, and the arbitrary scale constants can be chosen so that the separate scales match in the regions of overlap.[5]

In another paper [6] I introduced the following definition of an algebraic relation which approximates the pairwise discrimination structure. For $x,y \in T$, $x \gtrsim y$ if and only if $P(x,z) \geq P(y,z)$, for all $z \in T$. The relation $\gtrsim$ on T is referred to as the <u>trace</u> of the pairwise discrimination structure. It is easy to see that the trace is a transitive relation, but in general it need not be a weak order. That is, there may exist incomparable pairs (x,y) for which z and $z' \in T$ can be found such that $P(x,z) > P(y,z)$ and $P(x,z') < P(y,z')$. In [6] I found it necessary to suppose that this did not occur — that the trace is in fact a weak order. The following theorem gives a sufficient condition for this to be so.

<u>Theorem 4. Under the conditions of theorem 3, the trace of the pairwise discriminations forms a weak order and</u> v <u>is order preserving</u>.

Proof. By the definition of the trace, $x \gtrsim y$ is equivalent to $P(x,z) \geq P(y,z)$, for $z \in T$. But by theorem 3, this is equivalent to

5. The actual details of how this should best be done will very much depend upon the peculiar empirical difficulties of the several areas, and I do not intend to imply that a single general method will work. More experience in handling such problems exists in psychophysics than elsewhere, and some hint of one method used there is given in section 5.

$$\frac{1}{1+\frac{v(z)}{v(x)}} \geq \frac{1}{1+\frac{v(z)}{v(y)}} ,$$

which, by simple algebra, is equivalent to $v(x) \geq v(y)$.

Indeed, we have shown that $x \succsim y$ if and only if $P(x,y) \geq 1/2$. This, in turn, is known [3] to be equivalent to the condition of strong stochastic transitivity, namely,

if $P(x,y) \geq 1/2$ and $P(y,z) \geq 1/2$, then $P(x,z) \geq \max[P(x,y)P(y,z)]$.

4. Fechner's Problem

One way of phrasing theorem 3 is that axiom 1 is sufficient to make the discrimination problem mathematically one-dimensional. By this I mean that if we set $v(S) = \sum_{y \in S} v(y)$, then $P(x;S)$ depends only upon the ratio of $v(x)$ to $v(S)$. The one-dimensionality is particularly vivid for the pairwise discriminations where

$$P(x,y) = \frac{1}{1+\frac{v(y)}{v(x)}} .$$

The idea that discrimination along a single sensory continuum might be mathematically one-dimensional has long been common in psychology. It was first postulated by Fechner in psychophysics and it has been widely assumed, but without an axiomatic justification such as I have given here. As Fechner's assumption has been subject to a good deal of discussion and controversy in psychology, and as many psychologists now reject what is often called the Fechnerian position, I should like to examine what is involved in some detail.

It is generally held that Fechner assumed that the subjective sensation of intensity arising from physical stimuli which form a continuum is given by that transformation of the physical continuum

which renders discrimination dependent only upon sensation differences.[6] This is now believed on empirical grounds to have been wrong (see Stevens [15]). It seems to me that whether or not his assumption can be rejected greatly depends upon exactly what it is, and about this there is some confusion. As I see it, there are two quite distinct parts to it:

i. the probabilities of pairwise discriminations, the $P(x,y)$, are so constrained that there exists a real-valued mapping u of the stimuli and a function F of one real variable such that, for $P(x,y) \neq 0$ or 1,

$$P(x,y) = F\,[u(x)-u(y)],$$

and

ii. the function u of part i represents "subjective sensation".

Now, although part i must be true for part ii to have any meaning at all, the truth or falsity of part ii, however we may choose to interpret it, asserts nothing at all about the truth or falsity of part i. This simple point seems to have been slurred over a good deal in the discussions of Fechner's assumption(s).

Psychologists have interpreted part ii as implying various reasonable things about behavior, and these implications have turned out to be empirically false. For example, let x and y be two soft tones and x' and y' two loud tones, all of the same frequency, and such that $u(x)-u(y) = u(x')-u(y')$. It is argued that if u really represents subjective sensation, the two differences should seem to be of the same size to the subject; they do not. For such reasons the Fechnerian position has been rejected — not just part ii, but also part i. It would

6. Most often psychologists phrase Fechner's assumption in terms of the equality of sensation jnds and refer to the stated postulate as the principle that "equally often noticed differences are equal, unless always or never noticed." Of course, the jnd concept is actually an algebraic construct from statistical data, and it is not surprising to find that the two are actually the same assumption. A full discussion of this point will be found in [7].

appear that part i should be dealt with separately and, if true, retained, for the reduction of a multi-dimensional problem to a one-dimensional one is an achievement not to be lightly discarded.

Part of the reason for rejecting part i as well as part ii, even though the evidence does not force one to do so, is no doubt the fact that that restriction is difficult to accept as a primitive axiom. Somehow it is much too sophisticated and not sufficiently compelling to be treated other than as an interesting conjecture. What has been lacking is a basic axiom system from which it would appear as a consequence.

In axiom 1, however, we have a condition that is sufficient to prove Fechner's assumption i when discrimination is imperfect, and to do so quite generally without restricting U to be a continuum. This is easily seen by setting

$$u = \log v$$

in which case theorem 3 implies

$$P(x,y) = \frac{1}{1 + \frac{v(y)}{v(x)}} = \frac{1}{1 + \frac{e^{u(y)}}{e^{u(x)}}} = \frac{1}{1 + e^{-[u(x)-u(y)]}} .$$

For obvious reasons, I shall refer to log v as the <u>Fechnerian scale</u>.

It appears that v is a much more basic scale than Fechner's. For example, v enters in theorem 3 in a particularly simple way, making the calculation of P(x;S) almost trivial. In some of the following applications, v appears to play a more central role than log v. Nonetheless, if axiom 1 holds, Fechner was correct in the first half

of his assumption, though he need not have confined his conjecture to stimuli which form continua.

Recently, Stevens [15] has argued that, although it is true that discrimination is mathematically one dimensional, it depends upon ratios of scale values, not differences as assumed by Fechner. This is, of course, what we have shown must hold for the v scale; in addition, the results in the next section show other strong correspondences between our scale and the one Stevens has discussed.

If I am correct in my feeling that v is a basic scale, then it will need a name. There are a multitude of possibilities in the literature, among them response strength, sensation, value, and preference, but each is associated with a particular area of application and so would seem to limit the generality of the scale. But, as my wife pointed out, the initial letters of these names form a happy combination, and so I shall dub v the RSVP-scale.

5. Application to Psychophysics

One of the most important applications of this theory to psychophysics was given in the preceding section; however, as Fechner's problem is not confined to that case, I chose to treat it separately. There are, in addition, two other topics.

If, with the psychologists, we reject Fechner's second assumption that log v represents subjective sensation, then what does? Recently, Stevens [15] and Stevens and Galanter [16] have reviewed a large aggregate of data which, in part, seems to show that there are two quite distinct types of psychophysical scales. "Class I seems to include, among other things, those continue on which discrimination is mediated by an additive or prophetic process at the physiological level. An example is loudness, where we progress along the continuum by adding

excitation to excitation. Class II includes continua on which discrimination is mediated by a physiological process which is substitutive, or metathetic. An example is pitch, where we progress along the continuum by substituting excitation for excitation, i.e., by changing the locus of excitation." [15, p. 3].

In addition to this distinction by mechanism, there seem to be sharp behavioral differences between the two classes of scales. The properties of Class I seem to be more consistent and more thoroughly explored. For these, discrimination (pairwise) is approximately proportional to physical intensity (Weber's law), or more precisely (see Miller [9]) it is linear with intensity. Further, when a person is asked to assign numbers to stimuli so that the numbers correspond to subjective magnitudes (magnitude estimation), the data can be fitted quite accurately by a power function αx^{β}, where β is a constant between 0.3 and 2.0 depending upon the continuum and provided that it is measured in ordinary physical units (see Stevens [15]).

Let us suppose that axiom 1 holds and that the RSVP-scale v is a continuous function of physical intensity and that Class I continua are characterized by the property that the linear generalization of Weber's law is true, i.e., given any number k, $1 > k > 1/2$, there exist numbers $c(k)$ and $d(k)$ such that

$$P(x,y) = k \text{ if and only if } x = [1 + c(k)]y + d(k).$$

Then we show that

$$v(k) = \alpha[x + \gamma]^{\beta},$$

where $\alpha > 0$, $\beta = \dfrac{\log k - \log(1-k)}{\log[1 + c(k)]}$, and $\gamma = \dfrac{d(k)}{c(k)}$. Since, by theorem 3

$$P(x,y) = \frac{v(x)}{v(x) + v(y)},$$

the generalization of Weber's law can be written

$$v\left\{[1 + c(k)]y + d(k)\right\} = \frac{k}{1 - k} v(y).$$

By slightly modifying the results in [7], one can show that the solution to this equation is unique up to multiplication by a positive constant, and it is easy to show by substitution that the above v is this solution.

As far as mathematical form is concerned, the model leads to the correct result for Class I continua; however, the exponent β appears to be from one to two orders of magnitude larger than that obtained by direct methods. Stevens [15] reports $\beta = 0.3$ for loudness when y is measured in energy units. In a study of loudness discrimination of white noise (the results are fairly similar to those for pure tones), Miller [9] employed a technique in which the base stimulus was always present and periodically an increment of energy was added. He reports that for middle and high intensities, the Weber fraction (similar to c(k) above) corresponding to 50 per cent correct reports is 0.099 when y is measured in energy units. These data are not of the form needed in this model, for Miller did not use a forced choice technique — a failure to report an increment added is really an indifference report. If we suppose that in a forced choice situation half of these indifference reports would go one way and half the other — this is not strictly true, but, as will be shown, this will not affect the qualitative nature of the calculation — then $k = 0.75$ and $c(k) = 0.099$. Substituting in the above formula yields $\beta = 11.6$. Even if we took k as small as 0.6, our formula for β would yield 4.3, which is an order of magnitude larger than Steven's constant. I am quite uncertain as to how this discrepancy should be interpreted, but, as there can be no doubt that it is not an error of measurement, it bears some investigation.

One test of this model, which has not been available for earlier ones, is its prediction of the form of the discrimination functions. Once β is determined from k and $c(k)$, we predict that

$$P(x,y) = \frac{1}{1 + \left(\frac{y}{x}\right)^{\beta}} \ .$$

It is unclear exactly how the scales of Class II can be characterized, except that Weber's law does not hold and that magnitude estimation does not yield a power law. Once, however, a law of discrimination is given of the form

$$P(x,y) = k \quad \text{if and only if} \quad x = g(y),$$

theorem 3 and the analystic procedure given in [7] can be used to determine the RSVP-scale. For example, if there are any continua such that discrimination is independent of the physical value, i.e., $P(x,y) = k$ if and only if $x = y + c(k)$, then it is easy to show that

$$v(y) = \rho e^{\lambda y} \ ,$$

where $\rho > 0$ and $\lambda = \dfrac{\log k - \log(1-k)}{c(k)}$.

A second application of this theory is to Thurstone's [17] concept of a "discriminal process", which he introduced both as a possible explanation of imperfect discrimination and to arrive at certain mathematical relationships which might be observed. In its simplest form, the idea is to attach a density function $f(x,t)$ to each stimulus value x and to interpret it as characterizing the stimulus that the subject "thought" was administered. Thus, for example, when x and y are two sound energies, one supposes that an observation is drawn from $f(x,t)$ and another from $f(y,t)$ and whichever is the larger determines the stimulus that the subject calls louder. Note, we are assuming

independent observations. The model generalizes to any finite number of stimuli. To be more formal, let

$$F(x,t) = \int_{-\infty}^{t} f(x,\tau)d\tau.$$

The assumption is that

$$P(x;S) = \int_{-\infty}^{\infty} f(x,t) \prod_{y\varepsilon S-\{x\}} F(y,t)dt. \qquad (4)$$

This is the probability that, when S is the set of stimuli presented, x will be reported as loudest (or, more generally, largest on whatever dimension is being investigated). A similar expressinn can be written for the probability, say $P'(x;S)$, that x is reported least loud, namely,

$$P'(x;S) = \int_{-\infty}^{\infty} f(x,t) \prod_{y\varepsilon S-\{x\}} [1-F(y,t)]dt. \qquad (5)$$

The next theorem establishes that Thurstone's assumption of independent discriminal process is inconsistent with the assumption that both P and P' satisfy axiom 1.

Theorem 5. *Suppose that* $T = \{x,y,z\}$ *is a subset of the set* U *of positive real numbers such that, for all* $R \subset S \subset T$, $P(R;S)$ *and* $P'(R;S)$ *are defined and* $P(s,t)$ *and* $P'(s,t) \neq 0$ *or* 1, *for* $s,t \varepsilon T$. *If* P *and* P' *both satisfy axiom* 1, *then there do not exist density functions* $f(s,t)$ *for each* $s \varepsilon T$ *and* $t \varepsilon U$ *such that eqs.* 4 *and* 5 *hold for all* $S \subset T$.

Proof. Suppose the theorem is false, then

$$\begin{aligned} P(x;T)-P'(x;T) &= \int_{-\infty}^{\infty} f(x,t)\left\{F(y,t)F(z,t) - [1-F(y,t)][1-F(z,t)]\right\}dt \\ &= \int_{-\infty}^{\infty} f(x,t)[F(y,t) + F(z,T) - 1]dt \\ &= P(x,y) + P(x,z) - 1. \end{aligned}$$

By theorem 2,

$$P(x;T)-P'(x;T) = \frac{1}{1 + \frac{1-P(x,y)}{P(x,y)} + \frac{1-P(x,z)}{P(x,z)}} - \frac{1}{1 + \frac{P(x,y)}{1-P(x,y)} + \frac{P(x,z)}{1-P(x,z)}}$$

$$= [P(x,y)+P(x,z)-1]\left\{\frac{P(x,y)+P(x,z)-2P(x,y)P(x,z)}{[2-P(x,y)-P(x,z)+P(x,y)P(x,z)][1-P(x,y)P(x,z)]}\right\}.$$

Equating these two expressions, the term in braces must be 1, and that is easily reduced to the condition

$$[1-P(x,y)][1-P(x,z)][2-P(x,y)P(x,z)] = 0.$$

This can be satisfied only if either $P(x,y) = 1$ or $P(x,z) = 1$ or both, which is contrary to hypothesis.

6. Application to General Psychometric Problems

The most important implication of this theory for psychometric scaling is contained in theorem 3. So long as axiom 1 is satisfied, any set of alternatives has a numerical scale which is unique except for its unit. In particular, there is no need for the usual *ad hoc* normality and independence assumptions, and there is no basic difference between scaling a finite set of alternatives and scaling a psychophysical continuum.

In addition, there are two somewhat technical points that may be of methodological interest. A subject is sometimes asked to rank order stimuli according to some dimension instead of simply selecting the "largest" or the "smallest" stimulus. For our purpose, it is sufficient to suppose that he is asked to select his first and second choice and to indicate their ordering. Although it does not follow directly from axiom 1, the intuitive basis of that axiom would suggest that

$$R(x,y;T) = P(x;T)P(y;T-\{x\}) \tag{6}$$

gives the probability that x is the first choice and y the second. It is easy to see by axiom 1 that this is equivalent to

$$R(x,y;T) = P(y;T-\{x\})-P(y;T). \tag{7}$$

If, as before, P' refers to the probability of choosing the "smallest" stimuli, then $R'(x,y;T)$ can be defined similarly and it is interpreted as the probability that x is placed last and y next to last when such choices must be made from T.

Observe that when $T = \{x,y,z\}$ these two operations are tantamount to rank ordering the stimuli, and so one might suspect that $R(x,y;T)$ and $R'(z,y;T)$ would be equal. They are not generally, as we shall now show.

Theorem 6. *Suppose that* $T = \{x,y,z\}$ *and that, for all* $R \subset S \subset T$, $P(R;S)$ *and* $P'(R;S)$ *are defined and satisfy axiom* 1. *A necessary and sufficient condition for*

$$P(x;T)P(y,z) = P'(z;T)P(x,y)$$

is that $P(x,y) = P(y,z)$.

Proof. Replace $P(x;T)$ and $P'(z;T)$ by the expressions given in theorem 2 and simplify.

In words, if a person satisfies axiom 1 and if the probability of a ranking is given by eq. 6, then in general it matters whether he begins the ranking at the top or the bottom. This may not be unrelated to the fairly widespread, but so far as I know undocumented, impression that most people exhibit a characteristic direction of ordering, usually from the top down.

Our second point revolves around suggested devices for empirically estimating $P(x,y)$. The difficulty in making such estimates for many stimuli is that if the pair (x,y) is presented several times, one suspects that the first response is remembered and colors the answers to the later presentations. In other words, the first response alters the $P(x,y)$ governing the later responses. The problem, then, is to devise dodges which allow us to estimate $P(x,y)$ without actually presenting the simple choice between x and y more than once. One suggestion, which I first saw in [2], is to have the subject rank order several finite subsets of U. In [2], subsets of four elements were

were used. These sets can be chosen so that each pair of alternatives (x,y) appears a number of times, and an obvious estimate of $P(x,y)$ is the number of times that x is ranked superior to y divided by the total number of times the (x,y) pair appears. The following theorem justifies this procedure when subsets of three elements are used.

Theorem 7. Suppose that $T = \{x,y,z\}$ and that, for all $R \subset S \subset T$, $P(R;S)$ is defined and axiom 1 holds. Then,

$$P(x,y) = R(x,y;T) + R(x,z;T) + R(z,x;T).$$

Proof. By eq. 6,

$$\begin{aligned} R(x,y;T) + R(x,z;T) &= P(x;T)[P(y,z) + P(z,y)] \\ &= P(x;T). \end{aligned}$$

So, by eq. 7,

$$\begin{aligned} P(x,y) &= P(x;T-\{z\}) \\ &= R(z,x;T) + P(x;T) \\ &= R(z,x;T) + R(x,y;T) + R(x,z;T). \end{aligned}$$

Unfortunately, this result does not generalize to larger subsets. Suppose $T = \{w,x,y,z\}$ and define

$$R(w,x,y,z) = P(w;T)P(x;T-\{w\})P(y,z),$$

which we interpret as the probability of the rank order w,x,y,z. By writing down all the R expressions in which x precedes y and simplifying, the following estimate for $P(x,y)$ results:

$$\begin{aligned} P'(x,y) = P(x;T) &+ P(w;T)P(x;T-\{w\}) + P(z;T)P(x;T-\{z\}) \\ &+ P(x,y)[P(w;T)P(z;T-\{w\}) + P(z;T)P(w;T-\{z\})]. \end{aligned}$$

If axiom 1 holds, then in general $P'(x,y) \neq P(x,y)$. This can be shown by an example. Let

$$P(x,y) = 0.5, \quad P(x,z) = 0.7 \quad P(x,w) = 0.9$$

$$P(y,z) = 0.6, \quad P(y,w) = 0.8, \quad P(z,w) = 0.7.$$

From theorem 2, one calculates

$$P'(x,y) = 0.522 > 0.5 = P(x,y).$$

Since the discrepancy in this example (and in others like it) is small, P may actually serve as a fairly good estimate of P. This question needs further investigation.

7. Application to Stochastic Learning Theory

In the various stochastic models of learning [1], with which I must assume the reader is familiar at least in broad outline, it is assumed that the organism is repeatedly confronted by the same finite set T of alternatives, say $T = \{1,2,\ldots\ldots,i,\ldots r\}$. His choice on trial n is assumed to be determined by a probability distribution which we may denote by $\{P_n(i;T)\}$, and as a result of his choice an environmental event — an outcome — occurs which, if you please, can be thought of as rewarding or punishing the organism, thereby altering the probabilities which determine his choice on the next trial. We need not specify this any more fully than to say that for each alternative-outcome pair there is assumed to be an operator which transforms the probability distribution $\{P_n(i;T)\}$ into $\{P_{n+1}(i;T)\}$. The form of the operator will, in general, depend upon both the choice and the outcome, but it is assumed not to depend upon the trial number. Or put another way, it does not depend upon the previous history of the organism except to the extent that the history is summarized by the probability distribution on trial n. This is known as the "independence of path" assumption. Given that it is so, there is no loss of generality in suppressing the trial number n and simply denoting the probabilities of the present trial by $P(i;T)$ and those of the following trial by $P'(i;T)$.

Largely for reasons of mathematical simplicity, but also because of a rationale given by Estes (see chapter 2 of [1]) and because of

the combining of classes condition (see chapter 1 of [1]), the operators have generally been assumed to be linear functions of $P(i;T)$. Indeed, they can be written

$$P'(i;T) = \alpha P(i;T) + (1-\alpha)\lambda_i,$$

where α does not depend upon i. Most of the mathematical research has been devoted to determining some of the statistical properties of this linear model, mainly for T's having two elements, and applying it to learning data.

Most psychologists who have criticised this model have concentrated upon its questionable ability to handle certain empirical phenomena, and only to a lesser extent have they worried about its foundations. It has, however, been argued that the stimulus conditioning rationale for the linear operators is none too convincing, and the combining of classes condition has also been questioned. To me, one of the most frustrating features of this modern approach to learning, as distinct from some of the earlier theorizing, has been the apparent disparity in conception between it and the models that have been created to describe static choices, namely the psychophysical and utility models. Somehow, if there is in fact a mathematical structure to choice behavior, one is inclined to suppose that there should be something in common between the static and dynamic theories.

Intuitively, one connection suggests itself. The more traditional learning theorists (among others, Hull [4] and Spence [13]) have held that one should distinguish between the strength or intensity of a response and the observed likelihood of that response. For example, a 50-50 decision between two objectionable responses is not exactly the same as a 50-50 decision between two desirable ones. (Miller [10]). If one had a numerical measure of the strength of a response, then

this distinction might afford a basis for reconciling the stochastic models of learning and the ideas of psychophysical scaling and utility theory in such a way that a theory of the type desired by the behavior theorists would result. Although the behavior theorists, who have utilized the concept of response strength quite generally and amassed an impressive body of related empirical data, have attempted to cast these notions in a mathematical framework, no really satisfactory axiomatization has been given.

An alternative resolution of these two classes of theories is suggested by theorem 3. Since in a learning context it is not unreasonable to suppose that an alternative may be suddenly added or dropped, axiom 1 is not without meaning. Let us suppose that it is satisfied. Then, by theorem 3 we know that the RSVP-scale v exists and that

$$P(i;T) = \frac{v(i)}{\sum_{j \varepsilon T} v(j)} \; .$$

It is clear that if v is multiplied by a positive constant, the probability distribution is unchanged. So, if we are willing to emphasize the RS of the RSVP-scale and to identify it with the intuitive idea of response strength, the overall level of strength can change without necessarily altering the probability distribution. Of course, in a static model, I would suggest emphasizing the S, V, or P and would identify the scale with sensation, value, or preference, as the case may be.

Since the vector of RSVP-scale values, $\underset{\sim}{v} = [v(1), v(2), \ldots v(r)]$, uniquely determines the probability distribution, but not conversely, one is led along with the behavior theorists to suspect that the RSVP distribution may be more basic and that the learning model should be phrased in terms of changes in the RSVP-scale which indirectly alters the P distributions. Assuming that this is so and that the independence

of path assumption holds for the v's, though no longer necessarily for the P's, then a particular alternative-outcome pair will effect a change that can be represented by a (vector) operator of the form

$$\underset{\sim}{v}' = f(\underset{\sim}{v}) \tag{8}$$

There are two important constraints upon such operators. First, since the RSVP-scale is always non-negative, we have the

Non-negativeness condition

$$f(\underset{\sim}{v}) \geq \underset{\sim}{0}, \quad \text{for all } \underset{\sim}{v} \geq \underset{\sim}{0}, \tag{9}$$

where $\underset{\sim}{0}$ is the null vector with r components.

Second, since the unit for the RSVP-scale is not determined (theorem 3), it does not seem reasonable to permit the learning operator to depend upon the arbitrary unit chosen for calculations; hence we impose the

Invariance of unit condition

$$f(k\underset{\sim}{v}) = kf(\underset{\sim}{v}), \text{ for all real } k > 0 \text{ and all } \underset{\sim}{v} \geq \underset{\sim}{0} \tag{10}$$

These two conditions are insufficient to narrow the operators down to the point where it would be feasible to try to analyze data with this model, and so others must be added. Two directions suggest themselves, the first of which leads finally to the type of model that has been studied by Bush, Estes, and Mosteller. If we impose the

Superposition condition

$$f(\underset{\sim}{v} + \underset{\sim}{v}^*) = f(\underset{\sim}{v}) + f(\underset{\sim}{v}^*), \text{ for all } \underset{\sim}{v}, \underset{\sim}{v}^* \geq \underset{\sim}{0}, \tag{11}$$

then it is well known that this together with the invariance of unit condition constitutes the definition of a linear transformation in a vector space and that to each such there corresponds a matrix a_{ij} such that

$$v'(i) = \sum_{j=1}^{r} a_{ij} \, v(j)$$

By the non-negativeness condition (eq. 9), $a_{ij} \geq 0$ for $i,j \in T$.

Observe that, in general, $P'(i;T)$ cannot be expressed as a linear combination of the $P(j;T)$; however, if $[a_{ij}]$ satisfies the

Constant column sum condition

$$\sum_{i \in 1}^{r} a_{ij} = a, \quad \text{for all } j \in T, \tag{12}$$

then

$$P'(i;T) = \frac{\sum_{j=1}^{r} a_{ij} v(j)}{\sum_{i=1}^{r} \sum_{j=1}^{r} a_{ij} v(j)}$$

$$= \frac{\sum_{j=1}^{r} a_{ij} v(j)}{av(T)}$$

$$= \sum_{j=1}^{r} \frac{a_{ij}}{a} P(j;T)$$

The model defined by these four conditions will be referred to as the Alpha Model; it is the one that has been discussed uite fully by Bush and Mosteller in their book. To illustrate the relationship among parameters, let $r = 2$, then

$$P'(1,2) = \alpha P(1,2) + (1-\alpha)\lambda,$$

where

$$\alpha = (a_{11}-a_{12})/a \quad \text{and} \quad \lambda = a_{12}/(a_{12} + a_{21}).$$

Mathematically, the alpha model has the interesting feature that it is linear and satisfies the independence of path assumption both at the level of the RSVP-scale and at the level of the probability distribution.

The first of the two special conditions -- superposition -- is familiar and requires no discussion here. The second seems rather special, but it is needed if the change in the probability distribution

is to be expressed simply as a linear combination of the probabilities on the preceding trial. At the level of the RSVP-scale it has the following effect: each operator results in a v'-vector the sum of whose components is simply the constant a times the sum of the components of the v-vector.

The second direction that one can take is interesting because it leads to some new possibilities and because, to my mind, the assumptions needed seem fairly plausible. It will be recalled that in working with the alpha model Bush and Mosteller impose the combining classes condition which leads to the result that $P'(i;T)$ depends only upon $P(i;T)$, and not upon the rest of the distribution (for $r = 2$, this is trivially true). This conclusion has a certain intuitive appeal as a general condition on the learning operator, especially since it is another interpretation of the independence of irrelevant alternatives idea. So, we introduce the

Independence of Irrelevant Alternatives Condition. For each alternative-outcome pair, the learning operator f shall be of the form

$$v'(i) = f_i[v(i)], \text{ for } i \in T. \tag{13}$$

In the presence of this condition, the non-negativeness condition (eq. 9) becomes

$$f_i[v(i)] \geq 0, \quad \text{for all } v(i) \geq 0, \tag{14}$$

and the invariance of unit condition (eq. 10) becomes

$$f_i[kv(i)] = kf_i[v(i)], \quad \text{for all real } k > 0 \text{ and all } v(i) \geq 0. \tag{15}$$

Finally, it does not seem unreasonable to impose the

Continuity condition. Each f_i is a continuous function of its argument.

I shall refer to the learning model characterized by these four conditions (non-negativeness, invariance of unit, independence of irrelevant alternatives, and continuity) as the beta model. It is

well known that the only continuous solutions for eqs. 14 and 15 are of the form

$$f_i[v(i)] = \beta_i v(i),$$

where β_i is a non-negative constant. Observe that $\beta_i > 1$ effects an increase in $v(i)$ and $\beta_i < 1$ a decrease, so we may identify these with reward and non-reward (or punishment) of response i, if we so desire.

As is easily seen, the beta model is a special case of the matrix model (defined by non-negativeness, invariance of unit and superposition conditions) in which the matrix is diagonal.

An operator of the beta model yields a probability distribution of the form

$$P'(i;T) = \frac{\beta_i v(i)}{\sum_{j=1}^{r} \beta_j v(j)},$$

which we see cannot generally be expressed in terms of the $P(j;T)$ alone. There is, however, an important special case where some simplification is possible. Intuitively, it seems plausible that when alternative i is chosen, the effect of the outcome is to change $v(i)$ while leaving the other v's unaltered,

$$\beta_i = \beta$$

$$\beta_j = 1, \quad j \neq i.$$

This will be referred to as the <u>simple beta model</u>. It is easy to see that this implies that

$$P'(i;T) = \frac{\beta P(i;T)}{1 + (\beta-1)P(i;T)}$$

$$P'(j;T) = \frac{P(j;T)}{1 + (\beta-1)P(j;T)\frac{v(i)}{v(j)}}, \qquad j \neq i.$$

Although in general the probability distribution P' still does not depend upon the distribution P alone, when there are only two alternatives,

say i = 1 and j = 2, then the simple beta model becomes

$$P'(1,2) = \frac{\beta P(1,2)}{1 + (\beta-1)P(1,2)}$$

$$P'(2,1) = \frac{P(2,1)}{\beta - (\beta-1)P(2,1)}$$

$$= \frac{1-P(1,2)}{1 + (\beta-1)P(1,2)}$$

This is a non-linear operator of the type studied by Bush, Estes, Mosteller, and others.

The fact that the two alternative operator of the simple beta model is non-linear at the level of the P's should not cause alarm, for it is linear at the level of the v's. Furthermore, since there is no additive constant as in the usual linear operators of the alpha model, these operators commute, i.e., the order in which they are applied is immaterial. For some calculations, this results in a considerable simplification.

The real question now is whether the beta model can account for data as well as or better than the alpha model, and whether or not there are some learning experiments and phenomena that it can handle which had previously seemed outside the scope of stochastic learning theory. These are definitely open problems. At the time of writing, only some preliminary calculations on the simple beta model have been carried out (under the direction of Robert R. Bush). They were, however, sufficiently encouraging to make us undertake more detailed calculations; they will be reported elsewhere. Also, only a little thought has been given to phenomena that the beta model might be able to treat that have seemed to be outside the alpha model. This, of course, is a possibility since the new distribution of probabilities will generally depend upon the RSVP-scale, not just on the previous probabilities. Two ideas

immediately come to mind. First, predictions about behavior are possible when the set of alternatives is suddenly changed, as, for example, when the experimenter blocks a passage in a maze after a certain number of trials. Second, people have conjectured in the past that the time to reach a decision is highly dependent upon response strength, which suggests that one should study the relationship between latencies and distributions of RSVP-scale values. Such a study is in the planning stage.

8. Application to Choice Reaction Time

The following derivation of the distribution of decision latencies is well known. Denote by $P(0,t)$ the probability that a decision initiated at time 0 is made by time t, and set $Q(0,t) = 1-P(0,t)$. It is assumed that, if a decision has not been reached by time t, the probability that it is reached in the interval from t to $t+\Delta t$, where Δt is small, is given by $\lambda(t)\Delta t$. Thus,

$$Q(0,t+\Delta t) = Q(0,t)[1-\lambda(t)\Delta t],$$

so, rewriting,

$$\frac{Q(0,t+\Delta t)-Q(0,t)}{\Delta t} = -\lambda(t)Q(0,t).$$

Taking the limit, integrating, and entering in the reasonable initial condition $Q(0,0) = 1$,

$$Q(0,t) = \exp\left[-\int_0^t \lambda(\tau)d\tau\right].$$

Frequently it is assumed that

$$\lambda(t) = \begin{cases} \lambda & \text{for } t > t_o \\ 0 & \text{for } t \leq t_o \end{cases}$$

For reasons that are not entirely clear to me, I have always been somewhat uneasy about this derivation — what is actually assumed does not seem to be explicit. The following alternative derivation of the same result suggests that axiom 1, extended a bit, lies behind it.

Let us assume that axiom 1 holds for all (infinite, as well as finite) sets $R \subset S \subset T \subset U$ where the three probabilities $P(R;T)$, $P(S;T)$, and $P(R;S)$ are defined. In particular, suppose that U denotes the non-negative reals (the time continuum) and that $P(R;T)$ is always defined when R and T are intervals. Thus, if $\tau \leq t$ and

$$R = [x|\ \tau \leq x \leq t] = [\tau, t]$$
$$S = [x|\ 0 \leq x < \tau] = [0, \tau)$$
$$T = [x|\ 0 \leq x < \infty] = [0, \infty),$$

axiom 1 asserts that

$$Q([0,t];[0,\infty)) = Q([\tau,t];[\tau,\infty))Q([0,\tau);[0,\infty)).$$

Assuming that $Q([x,t];[x,\infty))$ is differentiable in t for every $x \geq 0$, take the logarithm of this expression and then differentiate with respect to t:

$$\frac{\partial Q([0,t];[0,\infty))}{\partial t} \Big/ Q([0,t];[0,\infty))$$
$$= \frac{\partial Q([\tau,t];[\tau,\infty))}{\partial t} \Big/ Q([\tau,t];[\tau,\infty))$$

Since this holds for every $\tau \geq 0$, the expression is only a function of t; call it $-\lambda(t)$. If we now integrate, we obtain

$$Q([\tau,t];[\tau,\infty)) = \exp\left[-\int_{-\infty}^{t} \lambda(x)dx + F(\tau)\right],$$

where F is an arbitrary function. However, if we impose the reasonable condition that

$$Q([\tau,\tau];[\tau,\infty)) = 1,$$

then we get

$$Q([\tau,t];[\tau,\infty)) = \exp\left[-\int_{\tau}^{t} \lambda(x)dx\right].$$

If, in addition, we assume that

$$Q([\tau,t];[\tau,\infty)) = Q([0,t-\tau];[0,\infty))$$

then it is easy to show that $\lambda(t)$ is a constant.

9. Application to Information Theory

The primary observation that I want to make in this section is that Shannon's axiomatic derivation [12] of entropy only makes sense if the device making the selections satisfies axiom 1. This means that whenever this statistic is used to describe animal or human behavior, either Shannon's axiomatic justification is implicitly rejected or axiom 1 is implicitly assumed.

The heart of the matter is contained in Shannon's third axiom defining entropy. He assumed that the entropy of a distribution $P(x;T)$, where T is finite, can always be expressed as the sum of two quantities, namely:

i. the entropy of the set T in which any subset S is treated as a single element having probability $P(S;T) = \sum_{x \varepsilon S} P(x;T)$, plus

ii. $P(S;T)$ times the entropy of the set S with the distribution $P(x;T)/P(S;T)$, for $x \varepsilon S$.

However, if we are discussing selections made by an organism, the probability of selections from S are independently defined as $P(x;S)$, and so the axiom only seems to make sense if we assume

$$P(x;S) = P(x;T)/P(S;T).$$

Rewriting and summing over all $x \varepsilon R$, where $R \subset S$, we obtain

$$P(R;T) = P(R;S)P(S;T),$$

which is axiom 1.

One trivial point of terminology. By theorem 3,

$$H = -\sum_{x \varepsilon T} P(x;T)\log P(x;T)$$

$$= \log v(T) - \sum_{x \varepsilon T} P(x;T)\log v(x),$$

so the entropy of the distribution $P(x;T)$ is the Fechnerian value of T minus the expected Fechnerian value of its elements.

10. Application to Utility Theory

As with the applications to psychological scaling, the central consequence for utility theory is contained in theorem 3: a utility function, unique except for its unit, exists over any set of goods provided that preference discrimination among the goods is not perfect and that axiom 1 is met. If, as may be generally expected, there are cases of perfect preference discrimination, the scale will have to be pieced together in the manner described following theorem 3. This may force one to introduce other goods, or some gambles among goods, in order that a sequence of overlapping sets exist connecting any two goods such that discrimination is imperfect within each of the sets.

In the remainder of this section, I should like to begin an examination of choice behavior among gambles in terms of the present theory. The problem is not resolved, and the results show that it is about as nasty as intuition suggests that it is.

Historically, gambles were introduced into the utility problem largely, though not entirely, to arrive at a cardinal (interval) utility scale, and as we now know they were not really necessary if imperfect discrimination is admitted. Nonetheless, the gambles generated from a set of goods and a (finite or infinite) set of events do form a possible set of alternatives of considerable importance. Especially significant for decision theory is whether a utility function exists having the property that the utility of a gamble equals the expected utility of its components. This so-called expected utility hypothesis follows from von Neumann and Morgenstern's [18] and related algebraic axiom systems. In another paper [6] I attempted to extend von Neumann and Morgenstern's algebraic model to a probabilistic one, and although interesting and fairly plausible results were obtained, it seemed clear

from a number of simple examples that something was not quite right. It turned out that if a utility function having the property described in theorem 4 above also satisfies the expected utility hypothesis, then it is a Fechnerian scale (which seems nice), but it also followed that there could be almost no cases of either perfect preference discrimination or perfect likelihood discrimination, and that just does not seem to be true of people. A major question in my mind has been whether to pursue that tack, assuming that cases of apparent perfect discrimination are not real but only result from the finite sizes of our samples, or whether to attack the problem anew assuming that both perfect and imperfect discriminations must be handled within the same theory. It seems as if a theory of the latter type will be quite complex. The following results suggest, however, that if axiom 1 is true we have no alternative.

As in [6], let A be a set (of pure alternatives) and E a Boolean algebra (of events) with null element o. A set S(A,E) (of gambles) is defined as follows:

i. if $a \in A$, then $a \in S(A,E)$, and

if $a,b \in S(A,E)$ and $\alpha \in E$, then $a\alpha b \in S(A,E)$;

where, for every $a,b \in A$ and $\alpha \in E$,

ii. $a\alpha a = a$,

iii. $aob = b$,

iv. $a\alpha b = b\bar{\alpha}a$, where $\bar{\alpha}$ denotes the complement of α.

The symbol $a\alpha b$ is interpreted as the gamble in which a is the outcome if the chance event α (e.g., rain tomorrow in New York City) occurs and b if it does not. Condition ii says that the "gamble" in which a is the outcome whether or not α occurs is exactly the same as the pure outcome a. The other two conditions can be interpreted equally simply.

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We shall suppose that choices from subsets of $S(A,E)$ are described by probabilities. The generic symbol P will be used, so, for example, $P(a\alpha b, c\beta d)$ denotes the probability that gamble $a\alpha b$ will be selected in preference to $c\beta d$ when only these two gambles are offered. In general, utility theory has confined itself to pairwise comparisons, but I shall of course suppose that the general probabilities of section 2 are defined. In addition, let us suppose that a similar set of probabilities, having the generic symbol Q, is defined over subsets of the set E of events. These are interpreted as the probability of selection in terms of (subjective) likelihood of occurence. Thus, for example, $Q(\alpha,\beta)$ denotes the (objective) probability that event α seems subjectively more likely to occur than event β.

In [6] the following axiom was introduced as a possible basic restriction in the study of choice behavior among gambles:

If $a,b \in A$ and $\alpha,\beta \in E$, then

$$P(a\alpha b, a\beta b) = P(a,b)Q(\alpha,\beta) + P(b,a)Q(\beta,\alpha).$$

We say that P is <u>decomposable</u> with respect to Q when this axiom is met. The intuitive grounds for supposing decomposability are these. In choosing between $a\alpha b$ and $a\beta b$, it seems reasonable to say that the former is preferred to the latter in exactly two cases:

i. when a seems preferable to b and α seems more likely to occur than β and

ii. when b seems preferable to a and β seems more likely to occur than α.

If we assume that the two acts of discrimination are statistically independent, which is at least plausible when a and b are pure alternatives, but not necessarily when they are themselves gambles, then the probability of the first occurence is $P(a,b)Q(\alpha,\beta)$ and of the latter is $P(b,a)Q(\beta,\alpha)$.

Since i and ii are exclusive alternatives, adding the two probabilities should yield the probability of choosing $a\alpha b$ over $a\beta b$, hence the axiom.

Since axiom 1 seems plausible within any choice context and decomposability is at least possible within the context of gambles, it would be interesting to know their joint consequences. I shall, however, confine my attention only to some special results which indicate an important necessary feature of a theory of gambles.

<u>Theorem</u> 8. <u>Suppose that</u>

i. P <u>is decomposable with respect to</u> Q,

ii. <u>for</u> P <u>over</u> $T = \{a\alpha b, a\beta b, a\gamma b\}$, <u>where</u> $a,b \in A$, <u>axiom</u> 1 <u>holds</u> <u>and</u> $P(s,t) \neq 0$ <u>or</u> 1 <u>for</u> $s,t \in T$,

iii. <u>for</u> Q <u>over</u> $F = \{\alpha, \beta, \gamma\}$ <u>axiom</u> 1 <u>holds and</u> $Q(\sigma,\tau) \neq 0$ <u>or</u> 1 <u>for</u> $\sigma,\tau \in F$.

<u>Then, either</u>

i. $P(a,b) = 0$, $1/2$ <u>or</u> 1, <u>or</u>

ii. $[Q(\alpha,\beta)-1/2] + [Q(\beta,\gamma)-1/2] = [Q(\alpha,\gamma)-1/2]$.

Proof. Suppose that $P(a,b) \neq 1$, then the decomposition hypothesis i can be written

$$P(a\alpha b, a\beta b) = P(b,a)\left\{1 + \left[\frac{P(a,b)}{P(b,a)} - 1\right] Q(\alpha,\beta)\right\}.$$

From hypothesis ii, we know (lemma 1) that

$$\frac{P(a\alpha b, a\beta b)}{P(a\beta b, a\alpha b)}\,\frac{P(a\beta b, a\gamma b)}{P(a\gamma b, a\beta b)} = \frac{P(a\alpha b, a\gamma b)}{P(a\gamma b, a\alpha b)}$$

Setting $A = \frac{P(a,b)}{P(b,a)} - 1$, these combine to yield

$$\frac{[1 + AQ(\alpha,\beta)][1 + AQ(\beta,\gamma)]}{[1 + AQ(\beta,\alpha)][1 + AQ(\gamma,\beta)]} = \frac{1 + AQ(\alpha,\gamma)}{1 + AQ(\gamma,\alpha)}$$

Crossmultiplying and simplifying yields

$$A(A+1)\left\{2[Q(\alpha,\beta) + Q(\beta,\gamma) - Q(\alpha,\gamma)] - 1\right\}$$
$$+ A^3[Q(\alpha,\beta)Q(\beta,\gamma)Q(\gamma,\alpha) - Q(\alpha,\gamma)Q(\gamma,\beta)Q(\beta,\alpha)] = 0.$$

From hypothesis iii, we know that lemma 1 holds for Q, so the coefficient of the A^3 term is 0; hence the first term is 0, which yields the assertion.

For any a and b, we should always be able to find three events α, β, and γ which are not perfectly discriminated and such that the three gambles $a\alpha b$, $a\beta b$, and $a\gamma b$ also are not perfectly discriminated. If so, then the theorem is tantamount to saying that axiom 1 and decomposability imply either that perfect discrimination exists among pure alternatives or an extremely special relation holds among the probabilities for the events. Since it seems unlikely that the latter relation will be sustained empirically (see the corollary below), one is forced to conclude that the pure alternatives must be perfectly discriminated. This seems to accord with one's intuitions about a lot of pure alternatives -- e.g., money — but it means that a choice theory for gambles is bound to be quite complex. I tried in [6] to avoid just that complexity by denying perfect discrimination among pure alternatives, and some of the results were a bit strange.

Corollary. Suppose that the conditions of the theorem hold for $F = \{\alpha, \bar{\alpha}, \beta, \bar{\beta}\}$ and $T = \{a\alpha b, a\bar{\alpha}b, a\beta b, a\bar{\beta}b\}$, that $Q(\beta,\bar{\beta}) = 1/2$ (i.e., β has subjective probability 1/2), and that $\frac{1}{2} < P(a,b) < 1$. Then, $Q(\alpha,\beta) \leq 3/4$.

Proof. By theorem 8,

$$Q(\alpha,\beta) + Q(\beta,\bar{\beta}) - Q(\alpha,\bar{\beta}) = 1/2$$

and

$$Q(\alpha,\beta) + Q(\beta,\bar{\alpha}) - Q(\alpha,\bar{\alpha}) = 1/2.$$

By theorem 2 of [6], $Q(\beta,\bar{\alpha}) = Q(\alpha,\bar{\beta})$. Substituting this and adding the two equations yields

$$\begin{aligned} 2Q(\alpha,\beta) &= 1 + Q(\alpha,\bar{\alpha}) - Q(\beta,\bar{\beta}) \\ &\leq 1 + 1 - 1/2 \\ &= 3/2. \end{aligned}$$

Even though I can cite no specific data, this result seems so counter-intuitive that one is forced to conclude that P(a,b) must equal 0, 1/2, or 1.

It will be recalled that in [6] a functional equation involving just Q was derived (theorem 7) from a notion of subjective independence (definitions 7 and 8). Using that equation and an argument much like that employed in theorem 8, one can show that, for any $\alpha \varepsilon E$ and the null event o, either $Q(\alpha,o) = 1/2$ or 1. In words, any event must either have the same subjective likelihood as the null event or invariably be seen as more likely to occur. Again, this seems in accord with one's intuition about such discriminations.

Although I find these results encouraging as to the correctness both of the decomposability axiom and of axiom 1, they do seem to discourage the hope that a simple, elegant probabilistic theory can be developed for the utility of gambles. The main problem will be to see in what, if any, sense the expected utility hypothesis is true within this framework. One thing is certain, it is not true for the RSVP-scale of theorem 3. This is suggested by theorem 10 of [6] where, under somewhat different conditions, it was shown that a utility function which is linear, i.e., satisfies the expected utility hypothesis, has to be a Fechnerian scale; and we can show it directly. I shall outline the proof without actually formally stating it. By v being linear, one means that there is a real-valued function ϕ on E such that

$$v(a\alpha b) = v(a)\phi(\alpha) + v(b)\phi(\bar{\alpha}).$$

From theorem 2 of [6] we know that

$$P(a\alpha b, a\beta b) = P(a\bar{\beta}b, a\bar{\alpha}b).$$

Substituting the expression from theorem 3 and simplifying we obtain

$$v(a\alpha b)v(a\bar{\alpha}b) = v(a\beta b)v(a\bar{\beta}b),$$

so, for fixed a and b, $v(a\alpha b)v(a\bar{\alpha}b)$ is independent of those α for which the hypothesis of theorem 3 holds. But, by linearity,

$$v(a\alpha b)v(a\bar{\alpha}b) = \frac{1}{4}\left\{[v(a)+v(b)]^2[\phi(\alpha)+\phi(\bar{\alpha})]^2 - [v(a)-v(b)]^2[\phi(\alpha)-\phi(\bar{\alpha})]^2\right\}$$

which is easily seen to depend upon α in general. So v cannot be linear.

11. Is the Basic Axiom True?

No doubt, this question has not been long out of mind, and I suspect that most readers have concluded that No, in general, it is not true. For instance, the following example, taken from [8], seems at first to cast doubt upon it.

"A gentleman wandering in a strange city at dinner time chances upon a modest restaurant which he enters uncertainly. The waiter informs him that there is no menu, but that this evening he may have either broiled salmon at \$2.50 or steak at \$4.00. In a first-rate restaurant his choice would have been steak, but considering his unknown surroundings and the different prices he elects the salmon. Soon after, the waiter returns from the kitchen, apologizes profusely, blaming the uncommunicative chef for omitting to tell him that fried snails and frog's legs are also on the bill of fare at \$4.50 each. It so happens that our hero detests them both and would always select salmon in preference to either, yet his response is 'Splendid, I'll change my order to steak.'"

Let us identify the following sets of alternatives:

$$T = \{\text{steak, salmon, snails, frog's legs}\}$$
$$S = \{\text{steak, salmon}\}$$
$$R = \{\text{salmon}\}.$$

The narrative certainly does not specify the exact probabilities of choice, but from the way it was phrased one suspects that they were not far from

$$P(R;T) = 0,\ P(R;S) > 0,\ \text{and}\ P(S;T) = 1,$$

in which case axiom 1 fails to hold.

Or does it? Are T and S actually the sets of alternatives from which the selections were made? Clearly not, for adding snails and frog's legs did not just increase the number of alternatives by two. They increased the diner's information as to exactly what the other two alternatives were, and therefore changed them. At first, he viewed his choice as being between "moderately expensive steak in a highly uncertain restaurant" and "less expensive salmon in the same restaurant". After the waiter returned, he was led to view it as being between "moderately expensive steak in a probably good restaurant" and "less expensive salmon in the same restaurant". In other words, the set of alternatives was totally changed. Since these sets of alternatives do not satisfy the inclusion relations in the hypothesis of axiom 1, there is no reason to expect it to hold.

Another, and an important, example where axiom 1 must generally fail is when a decision is made in two or more stages. Let us consider the two-stage case. A choice from T must be made. Suppose the subject partitions T into non-overlapping subsets $T_1, T_2, \ldots, T_t$, and first chooses one of these subsets and then an alternative from the chosen subset. It is easy to show that if axiom 1 holds for both component decisions, then it cannot usually hold for the overall choice from T. Of course, in general, it is only the overall choice that can be observed. From introspective and anecdotal evidence, one is reasonably certain that people sometimes decompose decisions in this manner, and it is tempting to suppose that animals do so too; however, there have been no methods for determining the categorizations used by non-humans, and so many important questions about the structuring of alternatives have remained unanswered.

One suggestion arises from this theory if the truth of axiom 1

is accepted for the component decisions. As stated above, axiom 1 does not hold in general for a two-stage decision, but it is easy to show that the equations

$$P(R;T) = P(R;T_i)P(T_i;T)$$

do hold for the subsets T_i forming the categorization. Thus, one may propose to use these equations to detect the categorizations from behavioral data. But this is an aside; I merely wanted to point out that data from such multi-stage decisions will lead to a rejection of axiom 1 if it is naively interpreted.

The point in both examples is, of course, that no general definition of the concept of an alternative has been given in the behavioral sciences. Each experimenter must make his own identification of alternatives for a specific empirical context in terms of that context. What we sorely need is a general theory of alternatives. To be sure, even without it, there is a remarkable amount of agreement as to what the alternatives are in simple situations; nonetheless, there have been many disagreements in the past. The question I raise is this: are there any apparent violations of axiom 1 that cannot be eliminated by a suitable and intuitively acceptable redefinition of the underlying set of alternatives?

If one is willing to conjecture the answer No, then we are led into an extremely subtle problem of scientific methodology and philosophy. If one can always save axiom 1 by a redefinition of the alternatives, then one is led to suggest that axiom 1 be accepted as correct and that it be used to determine the alternatives. Clearly, this is close to a form of insanity in which truth is by fiat, for why not choose any other relationship and set it up as a law, insisting that it is always correct and that other concepts must be changed to make it true. Close though

it may be, it is not always insane provided that certain other conditions are met. Certainly it has been done from time to time in physics with great success, and some philosophers have felt themselves forced to the position that some laws (e.g., conservation of energy) both possess empirical content and, at the same time, serve as organizing principles which suggest appropriate definitions in new areas of application.

The question, then, becomes the conditions under which it is not insane to make a statement of the form "for any choice situation, there exists a definition of the alternatives such that axiom 1 is true." I do not know whether philosophers have evolved such a list of conditions, but, as a practicing scientist, I think the following list will prove to be minimal in this case. First, for a wide variety of situations the axiom will have to be verified for carefully thought out, but independently given, definitions of the alternatives. By and large, these will probably be relatively simple situations. Second, in cases where the axiom appears to be violated, the required redefinition generally will have to result in intuitively acceptable insights into behavior. In many cases, one would expect the reaction "Of course, how did I miss that!" Third, the forced redefinition of the alternatives will have to be comparatively simple. Fourth, the axiom will have to have such rich and useful consequences in all fields of choice behavior when coupled with their particular laws that more will be lost by rejecting it than by keeping it. Put another way, it will have to be compatible with, or explain, the laws that have been established in special fields, and together they will have to explain a great deal of observed behavior.

In the preceding sections I have tried to show that axiom 1's range of application is fairly broad, but it still remains to see just how deep it actually goes. For example, does the suggested

modification of stochastic learning theory actually account for appreciably more learning phenomena than previous theories have? At present, the most one can say is that axiom 1 shows some promising, but highly inconclusive, symptoms of being a general law of choice behavior.

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